

# GLOBAL ESTIMATES FOR KERNELS OF NEUMANN SERIES AND GREEN'S FUNCTIONS

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ABSTRACT. We obtain global pointwise estimates for kernels of the resolvents  $(I - T)^{-1}$  of integral operators

$$Tf(x) = \int_{\Omega} K(x, y)f(y)d\omega(y)$$

on  $L^2(\Omega, \omega)$  under the assumptions that  $\|T\|_{L^2(\omega) \rightarrow L^2(\omega)} < 1$  and  $d(x, y) = 1/K(x, y)$  is a quasi-metric. Let  $K_1 = K$  and  $K_j(x, y) = \int_{\Omega} K_{j-1}(x, z)K(z, y) d\omega(z)$  for  $j \geq 1$ . Then

$$K(x, y)e^{cK_2(x, y)/K(x, y)} \leq \sum_{j=1}^{\infty} K_j(x, y) \leq K(x, y)e^{CK_2(x, y)/K(x, y)},$$

for some constants  $c, C > 0$ .

Our estimates yield matching bilateral bounds for Green's functions of the fractional Schrödinger operators  $(-\Delta)^{\alpha/2} - q$  with arbitrary nonnegative potentials  $q$  on  $\mathbb{R}^n$  for  $0 < \alpha < n$ , or on a bounded non-tangentially accessible domain  $\Omega$  for  $0 < \alpha \leq 2$ . In probabilistic language, these results can be reformulated as explicit bilateral bounds for the conditional gauge associated with Brownian motion or  $\alpha$ -stable Lévy processes.

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## 1. INTRODUCTION

This paper is dedicated to bounds for kernels of resolvents  $(I - T)^{-1}$  of integral operators

$$(1.1) \quad Tf(x) = \int_{\Omega} K(x, y)f(y) d\omega(y)$$

and their applications to estimates for Green's functions of Schrödinger operators and related quantities. Throughout,  $\omega$  is a positive measure on  $\Omega$ .

We consider the formal Neumann series

$$(I - T)^{-1} = I + \sum_{j=1}^{\infty} T^j$$

and the associated kernels  $K_1 = K$  and

$$(1.2) \quad K_j(x, y) = \int_{\Omega} K_{j-1}(x, z)K(z, y) d\omega(z)$$

for  $j \geq 2$ , of the operators  $T^j$ . Define the formal Green's function  $H : \Omega \times \Omega \rightarrow (0, +\infty]$  by

$$H(x, y) = \sum_{j=1}^{\infty} K_j(x, y).$$

Let  $\|T\| = \|T\|_{L^2(\omega) \rightarrow L^2(\omega)}$  denote the operator norm of  $T$  on  $L^2(\omega)$ .

We will consider the class of *quasi-metric* kernels, which have been considered previously in several papers, for example [KV] and [H]. A quasi-metric kernel  $K$  on a measure space  $(\Omega, \omega)$  is a measurable function from  $\Omega \times \Omega$  into  $(0, \infty]$  such that

(i)  $K$  is symmetric:  $K(x, y) = K(y, x)$  for all  $x, y \in \Omega$ ,

and

(ii)  $d = 1/K$  satisfies the quasi-triangle inequality

$$(1.3) \quad d(x, y) \leq \kappa(d(x, z) + d(z, y))$$

for all  $x, y, z \in \Omega$ , for some  $\kappa > 0$ , called the quasi-metric constant for  $K$ .

Our main theorem is the following.

**Theorem 1.1.** *Let  $(\Omega, \omega)$  be a  $\sigma$ -finite measure space. Let  $K$  be a quasi-metric kernel on  $\Omega$ . Suppose  $\|T\| < 1$ . Then there exists  $c = c(\kappa) > 0$  and  $C = C(\kappa, \|T\|) > 0$  such that*

$$(1.4) \quad K(x, y)e^{cK_2(x, y)/K(x, y)} \leq H(x, y) \leq K(x, y)e^{CK_2(x, y)/K(x, y)}.$$

It is well-known (see Lemma 2.1) that if  $\|T\| > 1$ , then  $H(x, y) = +\infty$  for all  $x$  and  $y$ . In the critical case  $\|T\| = 1$ , the lower bound still holds, but there are examples where  $H$  is finite a.e. and also examples where  $H = +\infty$  a.e., although  $K_2$  is finite a.e.

Kernels of the form  $K(x, y) = \sum_Q c_Q \chi_Q(x) \chi_Q(y)$ , where the sum is over all dyadic cubes in  $\mathbb{R}^n$ , were considered in [FV1], in connection with a discrete model of the Schrödinger equation (see Remark 2.8). Such kernels are quasi-metric with quasi-metric constant 1. Estimates of the form of inequality (1.4) were obtained in [FV1], under a Carleson condition on the sequence of scalars  $\{c_Q\}$ . (A sharp constant in the Carleson condition is established below; see Remark 2.8.)

In [FV2], estimate (1.4) and (1.7), (1.8) below were obtained under stronger assumptions.

Estimate (1.4) immediately extends (see Corollary 3.3) to the more general class of *quasi-metrically modifiable* kernels. A map  $K : \Omega \times \Omega \rightarrow (0, \infty]$  is quasi-metrically modifiable with constant  $\kappa$  if there exists a measurable function  $m : \Omega \rightarrow (0, \infty)$  such that  $\tilde{K}(x, y) = K(x, y)/(m(x)m(y))$  is a quasi-metric kernel on  $\Omega$  with quasi-metric constant  $\kappa$ . We call  $m$  a modifier of  $K$ .

Our main application is to the fractional Schrödinger operator

$$\mathcal{L}_\alpha = (-\Delta)^{\alpha/2} - q$$

with nonnegative potential  $q \in L^1_{loc}(\Omega)$  in some (possibly unbounded) domain  $\Omega \subseteq \mathbb{R}^n$ . Let  $G(x, y) = G^{(\alpha)}(x, y)$  be the Green's kernel associated with the fractional Laplacian  $(-\Delta)^{\alpha/2}$  on  $\Omega$  (see [L], [BBK], [H] for references and definitions). We note that  $G(x, y)$  is non-negative and symmetric on  $\mathbb{R}^n \times \mathbb{R}^n$ , and  $G(x, y) = 0$  if  $x \in (\overline{\Omega})^c$ ,  $y \in \mathbb{R}^n$ . For regular domains  $\Omega$ , this is true if  $x \in \Omega^c$ . For the sake of simplicity we will assume throughout the paper that domains  $\Omega$  are open and connected, so that  $G(x, y) > 0$  in  $\Omega \times \Omega$ , although most estimates remain true without the connectedness assumption.

By  $Gf$  we denote the corresponding Green potential operator, that is,

$$Gf(x) = \int_{\Omega} G(x, y) f(y) dy, \quad x \in \Omega.$$

For appropriate  $f$  and  $\Omega$ , we have  $Gf = 0$  in  $\Omega^c$  and  $(-\Delta)^{\alpha/2} Gf = f$  in  $\Omega$ . More generally,

$$G\mu(x) = \int_{\Omega} G(x, y) d\mu(y), \quad x \in \Omega,$$

where  $\mu$  is a Borel measure on  $\Omega$ .

Let  $q$  be a non-negative, locally integrable function on  $\Omega$ . Let

$$d\omega(x) = q(x)dx.$$

Let  $G_1 = G$  and define  $G_j$  inductively for  $j \geq 2$  by

$$G_j(x, y) = \int_{\Omega} G_{j-1}(x, z) G(z, y) d\omega(z).$$

The minimal Green's function associated with the fractional Schrödinger operator  $\mathcal{L}_{\alpha} = (-\Delta)^{\alpha/2} - q$  is

$$(1.5) \quad \mathcal{G}(x, y) = \sum_{j=1}^{\infty} G_j(x, y).$$

The corresponding Green's operator is

$$\mathcal{G}f(x) = \int_{\Omega} \mathcal{G}(x, y) f(y) dy.$$

Formally,  $u = \mathcal{G}f$  is the solution of the integral equation

$$u(x) = \int_{\Omega} G(x, y) u(y) d\omega(y) + Gf(x), \quad x \in \Omega, \quad a.e. \text{ in } \Omega,$$

and hence, by applying  $G$ , to the Schrödinger equation

$$\mathcal{L}_{\alpha} u = (-\Delta)^{\alpha/2} u - qu = f.$$

Theorem 1.2 below, which yields estimates for  $\mathcal{G}$  like those for  $H$  in Theorem 1.1, is applicable in the following cases.

(1) If  $\Omega = \mathbb{R}^n$  and  $0 < \alpha < n$ , then  $G$  is the classical Riesz kernel  $G^{(\alpha)}(x, y) = c_{n,\alpha} |x - y|^{\alpha-n}$ , which is a quasi-metric kernel.

(2) If  $\Omega$  is a ball or half-space, then for all  $0 < \alpha < n$ ,  $G = G^{(\alpha)}$  is a quasi-metrically modifiable kernel with modifier  $m(x) = \delta(x)^{\alpha/2}$ , where  $\delta(x)$  is the distance from  $x$  to the boundary  $\partial\Omega$ . This is easy to see from the concrete form of Green's kernel in these cases.

(3) If  $\Omega$  is a bounded domain with  $C^{1,1}$  boundary then

$$(1.6) \quad G^{(\alpha)}(x, y) \approx \frac{\delta(x)^{\alpha/2} \delta(y)^{\alpha/2}}{|x - y|^{n-\alpha} (|x - y| + \delta(x) + \delta(y))^{\alpha/2}},$$

where “ $\approx$ ” means that the ratio of the two sides is bounded above and below by positive constants depending only on  $\Omega$ , holds for  $0 < \alpha \leq 2$ ,  $\alpha < n$ . Hence  $G = G^{(\alpha)}$  is a quasi-metrically modifiable kernel with modifier  $m(x) = \delta(x)^{\alpha/2}$ .

(4) If  $\Omega$  is a bounded domain satisfying the boundary Harnack principle and  $0 < \alpha \leq 2$ , then  $G = G^{(\alpha)}$  is quasi-metrically modifiable with modifier  $m(x) = \min(1, G(x, x_0))$ , for  $x_0 \in \Omega$ , with a quasi-metric constant  $\kappa$  independent of  $x_0$ . In particular, this procedure is applicable when  $\Omega$  is a bounded Lipschitz domain, or more generally an NTA (non-tangentially accessible) domain. In fact, for  $0 < \alpha < 2$ , it suffices to assume that  $\Omega$  is merely an interior NTA domain which obeys the interior corkscrew condition. This class of  $\Omega$  coincides with the class of uniform (or  $\kappa$ -fat) domains. See [An], [BBK], [H], [K], [FV2], p. 118, for references and further discussion.

For the examples just listed, the following bilateral estimate follows immediately from the extension of Theorem 1.1 to the case of quasi-metrically modifiable kernels. Our upper estimate is new even in the classical case  $\alpha = 2$ ; the lower estimate is known for  $0 < \alpha \leq 2$  in the cases (1)-(4) discussed above (see [GH], and the literature cited there).

**Theorem 1.2.** *Let  $\Omega \subseteq \mathbb{R}^n$ ,  $n \geq 2$ . Assume that the Green's kernel  $G$  for  $(-\Delta)^{\alpha/2}$  on  $\Omega$  is quasi-metrically modifiable. Let  $q \in L^1_{loc}(\Omega)$  be non-negative, and set  $d\omega = qdx$ . Define  $\mathcal{G}$  by (1.5). Then there exists a positive constant  $c = c(\Omega, \alpha)$  such that*

$$(1.7) \quad \mathcal{G}(x, y) \geq G(x, y) e^{c G_2(x, y)/G(x, y)}.$$

*If, in addition,  $\|T\| < 1$ , where  $T$  is the operator*

*$Tf(x) = \int_{\Omega} G(x, y) f(y) d\omega(y)$ , then there exists a positive constant  $C = C(\Omega, \alpha, \|T\|)$  such that*

$$(1.8) \quad \mathcal{G}(x, y) \leq G(x, y) e^{C G_2(x, y)/G(x, y)}.$$

When  $\alpha = 2$ , there is a precise probabilistic formula

$$(1.9) \quad \mathcal{G}(x, y)/G(x, y) = E_{x, y} \left[ e^{\int_0^{\zeta} q(X_t) dt} \right],$$

where  $X_t$  is the Brownian path, with properly rescaled time, starting at  $x$ , and  $E_{x, y}$  is the conditional expectation conditioned on the event that  $X_t$  hits  $y$  before exiting  $\Omega$ , and  $\zeta$  is the time when  $X_t$  first hits  $y$ . The expression  $E_{x, y} \left[ e^{\int_0^{\zeta} q(X_t) dt} \right]$  is called the conditional gauge, or the

Feynman-Kac functional of the conditioned process (see [AS], [CZ]). Recently, similar formulas have been established in the case  $0 < \alpha < 2$  for the conditional gauge associated with an  $\alpha$ -stable Lévy process (see [BBK]). However, our approach is more general and covers even some cases with  $\alpha > 2$  (in particular,  $\Omega = \mathbb{R}^n$ ) for which there seems to be no probabilistic interpretation.

This probabilistic approach yields the lower bound (1.7) with  $c = 1$ , by applying Jensen's inequality in (1.9). On the other hand, the upper estimate (1.8), which can be rewritten as

$$(1.10) \quad E_{x,y} \left[ e^{\int_0^\zeta q(X_t) dt} \right] \leq e^{C E_{x,y} [\int_0^\zeta q(X_t) dt]},$$

seems to be new. It would be interesting to see if it has a probabilistic proof.

Some nonlinear analogues of Theorem 1.2 for quasilinear equations of the  $p$ -Laplace type with natural growth terms are obtained in [JV1], [JV2]. However, they are less precise and do not determine sharp constants in the conditions on  $q$ .

In Section 2, we prove Theorem 1.1. We discuss further results concerning integral operators with quasi-metric or quasi-metrically modifiable kernels in Section 3.

## 2. ESTIMATES FOR KERNELS OF NEUMANN SERIES

**Lemma 2.1.** *Let  $(\Omega, \omega)$  be a  $\sigma$ -finite measure space, and let  $K : \Omega \times \Omega \rightarrow (0, \infty]$  be a symmetric kernel on  $\Omega$ . Define  $T$  by (1.1). Let  $K_1 = K$  and define  $K_j$  by (1.2) for  $j \geq 2$ . If  $\|T\| > 1$ , then for every  $x, y \in \Omega$ ,*

$$H(x, y) = \sum_{j=1}^{\infty} K_j(x, y) = +\infty.$$

*Proof.* Suppose there exist  $x_0, y_0 \in \Omega$  such that  $H(x_0, y_0) < \infty$ . Then

$$\begin{aligned} \int_{\Omega} H(x_0, z) K(z, y_0) d\omega(z) &= \int_{\Omega} \sum_{j=1}^{\infty} K_j(x_0, z) K(z, y_0) d\omega(z) \\ &= \sum_{j=1}^{\infty} K_{j+1}(x_0, y_0) < H(x_0, y_0) < \infty. \end{aligned}$$

Since  $K(z, y_0) > 0$  for all  $z \in \Omega$ , we see that  $H(x_0, z) < \infty$  for a.e.  $z$ .

Let  $f(x) = H(x_0, x)$ . Then by the symmetry of  $H$ ,

$$\begin{aligned} Tf(x) &= \int_{\Omega} K(x, z)H(z, x_0) d\omega(z) = \sum_{j=1}^{\infty} \int_{\Omega} K(x, z)K_j(z, x_0) d\omega(z) \\ &= \sum_{j=1}^{\infty} K_{j+1}(x, x_0) < H(x, x_0) = H(x_0, x) = f(x). \end{aligned}$$

Since  $0 < f(x) < \infty$  a.e., Schur's Lemma implies that  $\|T\| \leq 1$ .  $\square$

We turn to the proof of the lower estimate for  $H$  in Theorem 1.1. It is only meaningful for  $\|T\| \leq 1$ , by the previous lemma, but the proof does not involve  $\|T\|$ .

We say that  $d$  is a quasi-metric with quasi-metric constant  $\kappa$ , on a nonempty set  $\Omega$ , if  $d : \Omega \times \Omega \rightarrow [0, \infty)$  is not identically 0 and satisfies  $d(x, y) = d(y, x)$  and the quasi-triangle inequality (1.3) for all  $x, y, z \in \Omega$ .

We will use the fact that  $\kappa \geq 1/2$ . To see this fact, select  $x \in \Omega$ . If  $d(x, x) > 0$ , applying (1.3) with  $x = y = z$  gives  $\kappa \geq 1/2$ . If  $d(x, x) = 0$ , then there must exist  $y \in \Omega$  such that  $d(x, y) > 0$ , and applying (1.3) with  $z = x$  implies that  $\kappa \geq 1$ . Note that  $\kappa = 1/2$  is attained in the case where  $d$  is constant.

**Lemma 2.2.** (*Ptolemy*) *Let  $d$  be a quasi-metric with constant  $\kappa$  on a set  $\Omega$ . Suppose  $y_1, y_2, y_3, y_4 \in \Omega$ . Let  $a = d(y_1, y_2), b = d(y_2, y_3), c = d(y_3, y_4), d = d(y_4, y_1), s = d(y_2, y_4)$ , and  $t = d(y_1, y_3)$ . Then*

$$st \leq 4\kappa^2 \max(ac, bd).$$

*Proof.* Without loss of generality, assume  $a = \min(a, b, c, d)$ . Then

$$s \leq \kappa(a + d) \leq 2\kappa d \text{ and } t \leq \kappa(a + b) \leq 2\kappa b.$$

Hence  $st \leq 4\kappa^2 bd$ .  $\square$

**Lemma 2.3.** *Let  $(\Omega, \omega)$  be a  $\sigma$ -finite measure space, and let  $K$  be a quasi-metric kernel on  $\Omega$ . Let  $K_1 = K$  and define  $K_j$  by (1.2) for  $j \geq 2$ . Then for  $c = (4\kappa^2)^{-1}$ ,*

$$(2.1) \quad \sum_{j=1}^{\infty} K_j(x, y) \geq K(x, y)e^{cK_2(x, y)/K(x, y)}.$$

*Proof.* Let  $d = 1/K$ . We can assume  $d(x, y) > 0$  for all  $x, y$ . To see this, for  $n \in \mathbb{N}$ , let  $K^{(n)} = \min(K, n)$ . Then  $K^{(n)}$  is a quasi-metric with the same quasi-metric constant as for  $K$ , corresponding to  $d_n = \max(d, 1/n)$ . Using (2.1) for  $K^{(n)}$  yields the result for  $K$ . Note that

at points where  $K(x, y) = \infty$  or  $K_2(x, y) = \infty$ , both sides of (2.1) are infinite.

Fix  $(x, y) \in \Omega$ . For  $z \in \Omega$ , define

$$F(z) = \frac{d(x, z)}{d(y, z)}.$$

For  $j \geq 2$ , let

$$A_j = \{(z_1, \dots, z_{j-1}) \in \Omega^{j-1} : F(z_1) \leq F(z_2) \leq \dots \leq F(z_{j-1})\}.$$

For  $z = (z_1, \dots, z_{j-1}) \in A_j$ , we have

$$d(x, z_{m+1})d(y, z_m) \geq d(x, z_m)d(y, z_{m+1})$$

for  $m = 1, \dots, j-2$ , and hence, by Lemma 2.2,

$$d(z_m, z_{m+1})d(x, y) \leq 4\kappa^2 d(x, z_{m+1})d(y, z_m).$$

Therefore, letting  $d\omega_{j-1}(z) = d\omega(z_1)d\omega(z_2) \cdots d\omega(z_{j-1})$ , we get

$$\begin{aligned} K_j(x, y) &= \int_{\Omega^{j-1}} \frac{1}{d(x, z_1)} \frac{1}{d(z_1, z_2)} \cdots \frac{1}{d(z_{j-2}, z_{j-1})} \frac{1}{d(z_{j-1}, y)} d\omega_{j-1}(z) \\ &\geq \left( \frac{d(x, y)}{4\kappa^2} \right)^{j-2} \int_{A_j} \frac{1}{d(x, z_1)} \frac{1}{d(z_1, y)} \frac{1}{d(x, z_2)} \frac{1}{d(z_2, y)} \cdots \\ &\quad \cdots \frac{1}{d(x, z_{j-1})} \frac{1}{d(z_{j-1}, y)} d\omega_{j-1}(z). \end{aligned}$$

This last integral is invariant under permutations of the indices  $1, \dots, j-1$  in the definition of  $A_j$ , and hence has value at least  $\frac{1}{(j-1)!}$  times the integral over all of  $\Omega^{j-1}$ , which splits and gives the value  $K_2(x, y)^{j-1}$ . Therefore

$$K_j(x, y) \geq c^{-1} K(x, y) \frac{(cK_2(x, y)/K(x, y))^{j-1}}{(j-1)!},$$

with  $c^{-1} = 4\kappa^2 \geq 1$ . We sum these estimates over  $j \geq 2$  and add  $K(x, y) = K_1(x, y)$  to obtain

$$\sum_{j=1}^{\infty} K_j(x, y) \geq K(x, y) e^{cK_2(x, y)/K(x, y)}.$$

□

Now we turn to the upper estimate of  $H$  in Theorem 1.1. The following lemma is standard (see [Hei], Proposition 14.5), except for the value of the constants, which we will use. As indicated in the proof in [Hei] (pp. 111-112), the inequality below holds with  $\beta \geq 2 \log_2(2\kappa)$  and  $C = (2\kappa)^2$ . Notice that in the proof in [Hei] the quasi-ultra-metric



condition  $d(x, y) \leq K \max[d(x, z), d(y, z)]$  is used in place of (1.3), so the constant  $K$  should be replaced with  $2\kappa$ .)

**Lemma 2.4.** [Hei] *Let  $d$  be a quasi-metric with constant  $\kappa$  on set  $\Omega$ . Then there exists a quasi-metric  $D$  with constant 1 such that*

$$(2.2) \quad D^\beta \leq d \leq CD^\beta$$

for  $\beta = 2 \log_2(2\kappa)$  and  $C = (2\kappa)^2$ .

Note that  $D$  may not be a metric because we may have  $D(x, x) > 0$  or  $D(x, y) = 0$  for  $x \neq y$ . The proof in [Hei] can easily be adapted to this case.

**Lemma 2.5.** (Inverse Ptolemy) *Let  $D$  be a quasi-metric with constant 1 on a set  $\Omega$ . Let  $y_1, y_2, y_3, y_4 \in \Omega$ . Let  $a = D(y_1, y_2)$ ,  $b = D(y_2, y_3)$ ,  $c = D(y_3, y_4)$ ,  $d = D(y_4, y_1)$ ,  $s = D(y_2, y_4)$ , and  $t = D(y_1, y_3)$ . Suppose the inequality*

$$ac \geq \tau^2 bd$$

holds with some  $\tau > 1$ . Then

$$st \geq (1 - \tau^{-1})^2 ac.$$

*Proof.* Without loss of generality,  $a \geq c$ . Since  $a^2 \geq ac \geq \tau^2 bd$ , either  $a \geq \tau b$  or  $a \geq \tau d$ . We can assume  $a \geq \tau b$ . Then

$$t \geq a - b \geq (1 - \tau^{-1})a.$$

Since  $ac \geq \tau^2 bd$ , either  $a \geq \tau d$  or  $c \geq \tau b$ . In the first case,

$$s \geq a - d \geq (1 - \tau^{-1})a \geq (1 - \tau^{-1})c.$$

In the second case,

$$s \geq c - b \geq (1 - \tau^{-1})c.$$

Hence we always have the estimate

$$st \geq (1 - \tau^{-1})a(1 - \tau^{-1})c = (1 - \tau^{-1})^2 ac.$$

□

**Corollary 2.6.** *Let  $d$  be a quasi-metric with constant  $\kappa$  on a set  $\Omega$ , and let  $D$  be the quasi-metric with constant 1 determined in Lemma 2.4. Let  $x, y, u, v \in \Omega$ . Suppose that for some  $\tau > 1$ ,*

$$\frac{D(x, v)}{D(y, v)} \geq \tau^2 \frac{D(x, u)}{D(y, u)}.$$

Then

$$d(x, y)d(u, v) \geq (1 - \tau^{-1})^{2\beta} (2\kappa)^{-4} d(x, v)d(y, u).$$

*Proof.* The result can be immediately obtained by raising both sides of the estimate in Lemma 2.5 to the power  $\beta$  and applying the bilateral inequality (2.2) for  $d$  and  $D^\beta$ .  $\square$

We now turn to the proof of the upper estimate in Theorem 1.1. Define the quasi-metric  $d = 1/K$ . Let  $D$  be the quasi-metric with constant 1 determined in Lemma 2.4. By considering  $\min(K, n)$  as in the proof of Lemma 2.3 and applying the monotone convergence theorem, we can assume  $D(x, y) > 0$  for all  $x, y \in \Omega$ . Fix  $x, y \in \Omega$  and define the function  $F$  by

$$(2.3) \quad F(z) = \frac{D(x, z)}{D(y, z)}, \quad z \in \Omega,$$

and  $f$  by

$$(2.4) \quad f(z) = \frac{1}{\sqrt{d(x, z)d(y, z)}}, \quad z \in \Omega.$$

Fix  $\tau > 1$  and  $j \geq 2$ . The heart of the proof is the following pointwise estimate.

**Lemma 2.7.** *Let  $\tau > 1$ , and let  $F$  and  $f$  be defined by (2.3) and (2.4) respectively. For every chain of points  $z_1, z_2, \dots, z_{j-1}$  in  $\Omega$ , there exists a subset*

$$M = \{m_1, m_2, \dots, m_\ell\} \subseteq \{1, \dots, j-2\},$$

with cardinality  $|M|$ , with  $m_1 < m_2 < \dots < m_\ell$  such that

$$(2.5) \quad F(z_{m_k}) < F(z_{m_{k+1}}) \text{ for all } k = 1, 2, \dots, \ell-1,$$

and

$$(2.6) \quad A \leq (2\kappa)^2 C(\tau, \kappa)^{|M|} \tau^{\beta(j-2-|M|)} d(x, y)^{|M|} B,$$

where

$$A = \frac{1}{d(x, z_1)} \frac{1}{d(z_1, z_2)} \cdots \frac{1}{d(z_{j-2}, z_{j-1})} \frac{1}{d(z_{j-1}, y)}$$

and

$$\begin{aligned} B &= f(z_1) \frac{1}{d(z_1, z_2)} \cdots \frac{1}{d(z_{m_1-1}, z_{m_1})} f(z_{m_1}) \\ &\times \prod_{k=1}^{\ell-1} \left[ f(z_{m_k+1}) \frac{1}{d(z_{m_k+1}, z_{m_k+2})} \cdots \frac{1}{d(z_{m_{k+1}-1}, z_{m_{k+1}})} f(z_{m_{k+1}}) \right] \\ &\times f(z_{m_\ell+1}) \frac{1}{d(z_{m_\ell+1}, z_{m_\ell+2})} \cdots \frac{1}{d(z_{j-2}, z_{j-1})} f(z_{j-1}). \end{aligned}$$

*Proof.* For  $m = 1, \dots, j-1$ , define

$$\Phi(m) = \min_{k \geq m} F(z_k).$$

Then  $\Phi$  is nondecreasing. Let

$$M = \{m \in \{1, 2, \dots, j-2\} : \Phi(m+1) \geq \tau^2 \Phi(m)\}.$$

Notice that  $F(z_m) \geq \Phi(m)$  and  $F(z_m) = \Phi(m)$  for  $m \in M$ . Hence  $F(z_m)$  is increasing for  $m \in M$ , because  $\Phi$  is increasing, so (2.5) holds. For  $m \in M$ ,

$$F(z_{m+1}) \geq \Phi(m+1) \geq \tau^2 F(z_m).$$

We have

$$\frac{A}{B} = \sqrt{\frac{d(y, z_1)}{d(x, z_1)}} \left[ \prod_{m \in M} \frac{\sqrt{d(x, z_m) d(y, z_m) d(x, z_{m+1}) d(y, z_{m+1})}}{d(z_m, z_{m+1})} \right] \sqrt{\frac{d(x, z_{j-1})}{d(y, z_{j-1})}}.$$

The estimate  $F(z_{m+1}) \geq \tau^2 F(z_m)$  means that the conditions of Corollary 2.6 hold for the points  $x, y, z_m, z_{m+1}$  for every  $m \in M$ . Thus

$$d(x, y) d(z_m, z_{m+1}) \geq (1 - \tau^{-1})^{2\beta} (2\kappa)^{-4} d(x, z_{m+1}) d(y, z_m).$$

Hence

$$\begin{aligned} \frac{A}{B} &\leq \left[ \frac{(2\kappa)^4}{(1 - \tau^{-1})^{2\beta}} \right]^{|M|} d(x, y)^{|M|} \\ &\quad \times \sqrt{\frac{d(y, z_1)}{d(x, z_1)}} \left[ \prod_{m \in M} \sqrt{\frac{d(x, z_m) d(y, z_{m+1})}{d(x, z_{m+1}) d(y, z_m)}} \right] \sqrt{\frac{d(x, z_{j-1})}{d(y, z_{j-1})}}. \end{aligned}$$

By the equivalence of  $d$  and  $D^\beta$ , we can estimate the last quantity by

$$\begin{aligned} \frac{A}{B} &\leq [(2\kappa)^2]^{|M|+1} \left[ \frac{(2\kappa)^4}{(1 - \tau^{-1})^{2\beta}} \right]^{|M|} d(x, y)^{|M|} \\ &\quad \times \left[ \frac{D(y, z_1)}{D(x, z_1)} \left( \prod_{m \in M} \frac{D(x, z_m) D(y, z_{m+1})}{D(x, z_{m+1}) D(y, z_m)} \right) \frac{D(x, z_{j-1})}{D(y, z_{j-1})} \right]^{\beta/2} \\ &= (2\kappa)^2 \left[ \frac{(2\kappa)^6}{(1 - \tau^{-1})^{2\beta}} \right]^{|M|} d(x, y)^{|M|} \left[ \frac{1}{F(z_1)} \left( \prod_{m \in M} \frac{F(z_m)}{F(z_{m+1})} \right) F(z_{j-1}) \right]^{\beta/2} \end{aligned}$$

Note that  $F(z_1) \geq \Phi(1)$ ,  $F(z_{j-1}) = \Phi(j-1)$ , and recall that for every  $m \in M$ , we have  $F(z_m) = \Phi(m)$  and  $F(z_{m+1}) \geq \Phi(m+1)$ . Hence we can estimate the product

$$\frac{1}{F(z_1)} \left( \prod_{m \in M} \frac{F(z_m)}{F(z_{m+1})} \right) F(z_{j-1})$$

by

$$\frac{1}{\Phi(1)} \left( \prod_{m \in M} \frac{\Phi(m)}{\Phi(m+1)} \right) \Phi(j-1) = \frac{\Phi(m_1)}{\Phi(1)} \frac{\Phi(m_2)}{\Phi(m_1+1)} \cdots \frac{\Phi(j-1)}{\Phi(m_\ell+1)}.$$

We now observe that the inequality  $\Phi(m+1) \leq \tau^2 \Phi(m)$  holds for every  $m \in [1, m_1-1] \cup [m_1+1, m_2-1] \cup \cdots \cup [m_\ell+1, j-1]$ , i.e., for  $m \notin M$ , so  $\Phi(m_1) \leq \tau^{2(m_1-1)} \Phi(1)$ ,  $\Phi(m_2) \leq \tau^{2(m_2-m_1-1)} \Phi(m_1+1)$ , and so on up to  $\Phi(j-1) \leq \tau^{2(j-1-m_\ell-1)} \Phi(m_\ell+1)$ . Therefore the last product does not exceed  $\tau^{2(j-2-|M|)}$ . Combining these estimates we get the conclusion of the lemma with  $C(\tau, \kappa) = (2\kappa)^6 (1 - \tau^{-1})^{-2\beta}$ .  $\square$

PROOF OF THEOREM 1.1. Let  $F$  and  $f$  be defined by (2.3) and (2.4) respectively. Integrating the estimate (2.6) with respect to  $\omega_{j-1}$  and summing over all possible choices of  $M$ , we arrive at the inequality

$$K_j(x, y) \leq (2\kappa)^2 \sum_{\ell=0}^{j-2} C(\tau, \kappa)^\ell \tau^{\beta(j-1-\ell)} d(x, y)^\ell \\ \times \sum_{1 \leq m_1 < m_2 < \cdots < m_\ell \leq j-2} I_j(m_1, m_2, \dots, m_\ell),$$

for  $j \geq 2$ , where

$$I_j(m_1, m_2, \dots, m_\ell) \\ = \int_{\{F(z_{m_1}) < F(z_{m_2}) < \cdots < F(z_{m_\ell})\}} \\ (fT^{m_1-1}f)(z_{m_1}) \left( \prod_{k=1}^{\ell-1} (fT^{m_{k+1}-m_k-1}f)(z_{m_{k+1}}) \right) (fT^{j-1-m_\ell-1}f)(z_{j-1}) \\ d\omega(z_{m_1}) \cdots d\omega(z_{m_\ell}) d\omega(z_{j-1}).$$

Let  $\alpha \in (1, \|T\|^{-1})$ . Define  $S = \sum_{k \geq 0} \alpha^k T^k$ . Since  $f \geq 0$ , we have the pointwise inequality

$$T^k f \leq \alpha^{-k} S f$$

for all  $k \geq 0$ . Thus

$$I_j(m_1, \dots, m_\ell) \leq \alpha^{-(j-2-\ell)} \int_{\{F(z_{m_1}) < F(z_{m_2}) < \cdots < F(z_{m_\ell})\}} \\ g(z_{m_1}) \cdots g(z_{m_\ell}) g(z_{j-1}) d\omega(z_{m_1}) \cdots d\omega(z_{m_\ell}) d\omega(z_{j-1}) \\ \leq \frac{\alpha^{-(j-2-\ell)}}{\ell!} \|g\|_{L^1(\omega)}^{\ell+1},$$

where  $g = fSf$ . The last inequality is true because the integral is invariant with respect to the permutations of  $m_1, \dots, m_\ell$  in the domain of integration.

Note that

$$\|g\|_{L^1(\omega)} \leq \|S\| \|f\|_{L^2(\omega)}^2 = \|S\| K_2(x, y).$$

Putting these estimates together, we obtain

$$K_j(x, y) \leq (2\kappa)^2 K_2(x, y) \sum_{\ell=0}^{j-2} \binom{j-2}{\ell} (\tau^\beta \alpha^{-1})^{j-2-\ell} \frac{C(\tau, \kappa)^\ell}{\ell!} \left( \frac{K_2(x, y)}{K(x, y)} \right)^\ell.$$

If  $0 < \rho < 1$ , then for each  $\ell \geq 0$ ,

$$(2.7) \quad \sum_{j=\ell}^{\infty} \binom{j}{\ell} \rho^{j-\ell} = \frac{1}{(1-\rho)^{\ell+1}},$$

by differentiating  $(1-\rho)^{-1} = \sum_{j=0}^{\infty} \rho^j$  a total of  $\ell$  times. Select  $\tau > 1$  such that  $\rho \equiv \tau^\beta \alpha^{-1} < 1$ . Using (2.7),

$$\sum_{j=2}^{\infty} K_j(x, y) \leq \frac{(2\kappa)^2}{1-\tau^\beta \alpha^{-1}} K_2(x, y) \exp \left( \frac{C(\tau, \kappa)}{1-\tau^\beta \alpha^{-1}} \frac{K_2(x, y)}{K(x, y)} \right).$$

To complete the proof of the upper bound in (1.4), it remains only to use the elementary inequality  $1 + CVe^{CV} \leq e^{2CV}$ , valid for all  $C, V > 0$ .

□

**Remark 2.8.** Theorem 1.1 is applicable to the discrete model of the Schrödinger equation considered in [FV1]. Let  $\omega$  be a Borel measure on  $\mathbb{R}^n$ , and let  $\mathcal{Q}$  denote the family of dyadic cubes in  $\mathbb{R}^n$ . For a sequence  $s = \{s_Q\}_{Q \in \mathcal{Q}}$  of positive scalars, we consider an operator  $T$  defined by (1.1) with kernel

$$K(x, y) = \sum_{Q \in \mathcal{Q}} \frac{s_Q}{\omega(Q)} \chi_Q(x) \chi_Q(y),$$

where the sum is taken over all dyadic cubes  $Q$  such that  $\omega(Q) \neq 0$ . This is a quasi-metric kernel with constant  $\kappa = 1$  (moreover,  $d(x, y) = 1/K(x, y)$  is an ultra-metric; that is,  $d(x, y) \leq \max(d(x, z), d(z, y))$ ). Note that, for  $g \in L^2(\omega)$ ,

$$\langle Tg, g \rangle = \sum_{Q \in \mathcal{Q}} \frac{s_Q}{\omega(Q)} \left( \int_Q g d\omega \right)^2 \leq \|T\| \cdot \|g\|_{L^2(\omega)}^2.$$

Define the discrete Carleson norm of  $s = \{s_Q\}_{Q \in \mathcal{Q}}$  by

$$\|s\|_\omega = \sup_{Q \in \mathcal{Q}} \omega(Q)^{-1} \sum_{P \in \mathcal{Q}: P \subseteq Q} s_P \omega(P).$$

Then  $\|s\|_\omega \leq \|T\| \leq 4\|s\|_\omega$ , where the constant 4 is sharp (see [NTV], Theorem 3.3). Consequently, by Theorem 1.1 estimate (1.4) holds if  $\|s\|_\omega < \frac{1}{4}$ , where the constant  $\frac{1}{4}$  is sharp as well. Indeed, (1.4) yields  $\|T\| \leq 1$  by Schur's lemma, and so  $\frac{1}{4}$  cannot be replaced by any larger constant in view of Theorem 3.3 in [NTV].

Such estimates of the corresponding Green's function were obtained earlier in [FV1] by a different method, with  $\frac{1}{12}$  in place of  $\frac{1}{4}$ , along with estimates of solutions to the discrete Schrödinger equation  $u = Tu + f$ .

### 3. FURTHER RESULTS ON QUASI-METRIC AND QUASI-METRICALLY MODIFIABLE KERNELS

Let  $T$  be defined by (1.1) where  $K : \Omega \times \Omega \rightarrow (0, +\infty]$  is a non-negative kernel. The minimal positive solution  $u_0$  of the equation  $u = Tu + 1$  is obviously given by  $u_0 = 1 + \sum_{j=1}^{\infty} T^j 1$ . Our next result is a bilateral pointwise estimate of  $u_0$ . In the case where  $K$  is the Green's function  $G$  of  $(-\Delta)^{\alpha/2}$ , the function  $u_0 = \mathcal{G}1$  is of interest in the study of Schrödinger equations.

**Theorem 3.1.** *Let  $(\Omega, \omega)$  be a  $\sigma$ -finite measure space. Suppose that  $K$  is a quasi-metric kernel with constant  $\kappa$  on  $(\Omega, \omega)$ , and  $T$  is the corresponding integral operator. Then there exists  $c = c(\kappa) > 0$  such that the minimal positive solution  $u_0$  of the equation  $u = Tu + 1$  satisfies*

$$(3.1) \quad u_0 \geq e^{cT^1}.$$

*If  $\|T\| < 1$ , then there exists  $C = C(\kappa, \|T\|) > 0$  such that*

$$(3.2) \quad u_0 \leq e^{CT^1}.$$

*Proof.* We first consider the case where  $\Omega$  is bounded with respect to  $d = 1/K$ , that is, when  $D = \sup_{x, y \in \Omega} d(x, y) < +\infty$ . We will add a point  $z$  to  $\Omega$  which is far away from all other points. That is, we choose  $z \notin \Omega$  and consider the space  $\Omega^* = \Omega \cup \{z\}$  with quasi-metric  $d^*$  defined by  $d^*(x, y) = d(x, y)$  if  $x, y \in \Omega$ ,  $d^*(x, z) = d^*(z, x) = D$  for all  $x \in \Omega$ , and  $d(z, z) = 0$ . Then  $d^*$  is a quasi-metric on  $\Omega^*$  with quasi-metric constant  $\kappa^* = \max(\kappa, 1)$ . We also extend  $\omega$  to a measure  $\omega^*$  on  $\Omega^*$  by setting  $\omega^*|_\Omega = \omega$ , and  $\omega^*(\{z\}) = 0$ .

Note that the iterates  $K_j^*$  of  $K^* = 1/d^*$  with respect to  $\omega^*$  agree with the iterates  $K_j$  of  $K$  with respect to  $\omega$  on  $\Omega \times \Omega$  since  $\omega^*(\{z\}) = 0$ ,

and that the norm of the integral operator with the kernel  $K^*$  on  $(\Omega^*, \omega^*)$  is the same as  $\|T\|$ .

For all  $x \in \Omega$ ,

$$\frac{K_2^*(x, z)}{K^*(x, z)} = D \int_{\Omega^*} K^*(x, y) K^*(y, z) d\omega^*(y) = \int_{\Omega} K(x, y) d\omega(y) = T1(x)$$

and

$$\begin{aligned} u_0(x) &= 1 + \sum_{j=1}^{\infty} \int_{\Omega} K_j(x, y) d\omega(y) = 1 + D \sum_{j=1}^{\infty} \int_{\Omega^*} K_j^*(x, y) K^*(y, z) d\omega^*(y) \\ &= DK^*(x, z) + D \sum_{j=2}^{\infty} K_j^*(x, z) = D \sum_{j=1}^{\infty} K_j^*(x, z). \end{aligned}$$

Hence, applying the lower estimate in Theorem 1.1 on the space  $\Omega^*$  we get, for all  $x \in \Omega$ ,

$$(3.3) \quad u_0(x) \geq DK^*(x, z) e^{cK_2^*(x, z)/K^*(x, z)} = e^{cT1(x)}.$$

Similarly, the upper estimate in Theorem 1.1 gives, for all  $x \in \Omega$ ,

$$(3.4) \quad u_0(x) \leq DK^*(x, z) e^{CK_2^*(x, z)/K^*(x, z)} = e^{CT1(x)}.$$

For  $\Omega$  not bounded with respect to  $d$ , select  $x_0 \in \Omega$  and let  $\Omega_n = \{x \in \Omega : d(x, x_0) < n\}$ . Let  $\omega_n$  be the restriction of  $\omega$  to  $\Omega_n$ , and let  $d_n$  and  $K^{(n)}$  be the restrictions of  $d$  and  $K$  to  $\Omega_n \times \Omega_n$  respectively. Then  $K^{(n)}$  is a quasi-metric kernel on  $\Omega_n$ . The corresponding integral operator  $T_n$  defined by

$$T_n f(x) = \int_{\Omega_n} K^{(n)}(x, y) f(y) d\omega_n(y) = \int_{\Omega} K(x, y) \chi_{\Omega_n}(y) d\omega(y)$$

satisfies

$$\|T_n\|_{L^2(\Omega_n) \rightarrow L^2(\Omega_n)} \leq \|T\|_{L^2(\Omega) \rightarrow L^2(\Omega)},$$

and  $T_n 1 \rightarrow T1$  pointwise as  $n \rightarrow \infty$ .

Let  $T_n^j$  be the  $j^{th}$  iterate of  $T_n$  and let  $K_j^{(n)}$  be the kernel of  $T_n^j$ . Then  $K_j^{(n)}(x, y)$  is non-decreasing in  $n$  and converges to  $K_j(x, y)$  pointwise as  $n \rightarrow \infty$ , for each  $j \in \mathbb{N}$ , by the monotone convergence theorem. Let  $u_0^{(n)} = 1 + \sum_{j=1}^{\infty} T_n^j 1$ . By the monotone convergence theorem,  $u_0^{(n)} \rightarrow u_0$  pointwise as  $n \rightarrow \infty$ .

Applying the estimates for the bounded space  $\Omega_n$ , and passing to the limit as  $n \rightarrow \infty$ , we see that estimates (3.1) and (3.2) hold in the unbounded case as well.  $\square$

We now turn to characterizing the kernels for which  $\sum_{j=1}^{\infty} K_j(x, y)$  is pointwise equivalent to  $K(x, y)$ .

**Theorem 3.2.** *Let  $(\Omega, \omega)$  be a  $\sigma$ -finite measure space. Suppose  $K : \Omega \times \Omega \rightarrow (0, +\infty]$  is a quasi-metric kernel with constant  $\kappa$ , and that  $K$  is not identically  $\infty$ . Let  $T$  be the integral operator corresponding to  $K$  and let  $u_0$  be the minimal positive solution of the equation  $u = Tu + 1$ . Then the following statements are equivalent:*

(a) *There exists  $C_1 > 0$  such that  $\sum_{j=1}^{\infty} K_j(x, y) \leq C_1 K(x, y)$  for all  $x, y \in \Omega$ .*

(b)  *$\|T\| < 1$  and  $K_2(x, y) \leq C_2 K(x, y)$  for all  $x, y \in \Omega$ , for some  $C_2 > 0$  (or, equivalently,  $\sup_{x \in \Omega} T1(x) < +\infty$ ).*

(c)  *$\sup_{x \in \Omega} u_0(x) < +\infty$ .*

*Proof.* We first show that the condition  $K_2 \leq C_2 K$  is equivalent to the boundedness of  $T1$ . Notice that by the quasi-metric property of  $K$ ,

$$K(x, z)K(y, z) \leq \kappa K(x, y)[K(x, z) + K(y, z)].$$

Hence,

$$\begin{aligned} K_2(x, y) &= \int_{\Omega} K(x, z)K(y, z)d\omega(z) \\ &\leq \kappa K(x, y) \int_{\Omega} K(x, z)d\omega(z) + \kappa K(x, y) \int_{\Omega} K(y, z)d\omega(z) \\ &= \kappa K(x, y)[T1(x) + T1(y)], \end{aligned}$$

so that the boundedness of  $T1$  implies that  $K_2 \leq C_2 K$ .

Now suppose  $K_2 \leq C_2 K$ . Fix  $x \in \Omega$ . Suppose first that

(1)  $0 < \sup_{y \in \Omega} d(x, y) = D < +\infty$ . Pick any  $y \in \Omega$  with  $d(x, y) > \frac{D}{2}$ . Then

$$d(y, z) \leq \kappa[d(x, y) + d(x, z)] \leq 3\kappa d(x, y)$$

for all  $z \in \Omega$ , and so  $K(y, z) \geq \frac{1}{3\kappa} K(x, y)$ . It follows that

$$K_2(x, y) = \int_{\Omega} K(x, z)K(y, z)d\omega(z) \geq \frac{1}{3\kappa} K(x, y)T1(x),$$

whence  $T1(x) \leq 3\kappa C_2$ .

Now suppose that

(2)  $\sup_{y \in \Omega} d(x, y) = +\infty$ . Then there is a sequence  $y_n \in \Omega$  such that  $0 < r_n = d(x, y_n) \rightarrow +\infty$  as  $n \rightarrow \infty$ . For every  $z \in B(x, r_n)$ , we have

$$d(y_n, z) \leq \kappa[d(x, y_n) + d(x, z)] \leq 2\kappa d(x, y_n).$$



Hence,

$$\begin{aligned} K_2(x, y_n) &\geq \int_{B(x, r_n)} K(x, z)K(y_n, z)d\omega(z) \\ &\geq \frac{1}{2\kappa}K(x, y_n) \int_{B(x, r_n)} K(x, z)d\omega(z), \end{aligned}$$

and consequently  $\int_{B(x, r_n)} K(x, z)d\omega(z) \leq 2\kappa C_2$  for all  $n \in \mathbb{N}$ . Passing to the limit as  $n \rightarrow \infty$ , we get  $T1(x) \leq 2\kappa C_2$ .

Now Theorem 1.1 shows that (b) implies (a), and Theorem 3.1 shows that (b) implies (c).

If  $u_0$  is bounded by  $C$ , then by Theorem 3.1,  $T1$  is bounded. From  $u_0 = Tu_0 + 1$ , we obtain

$$Tu_0 = u_0 - 1 \leq \left(1 - \frac{1}{C}\right)u_0.$$

Hence  $\|T\| \leq 1 - 1/C < 1$ , by Schur's Lemma. So (c) implies (b).

It remains to show that (a) implies (b). Condition (a) trivially implies  $K_2 \leq C_2 K$ , and hence we have that  $T1$  is bounded. It remains to show that (a) implies  $\|T\| < 1$ . Since the kernel of  $T$  is positive,  $T$  is bounded on  $L^\infty(\omega)$ , and by duality on  $L^1(\omega)$ . Thus by interpolation  $T$  is a bounded operator on  $L^2(\omega)$ . Comparing kernels and using (a), there exists  $C$  so that for all  $n$ ,

$$\|T^n\| \leq \|T + T^2 + \cdots + T^n\| \leq C.$$

Since the kernel of  $T$  is symmetric,  $T$  is self-adjoint, so  $\|T\|$  coincides with the spectral radius  $r(T)$  on  $L^2(\omega)$ . Hence it suffices to show that  $\|T^n\| < 1$  for some  $n$ . If not, given  $n$  we can select  $f$  such that  $\|f\| = 1$  and  $\|T^n f\| > 1/2$ . We can assume  $f \geq 0$ . Then for all  $m \leq n$

$$\|T^m f\| \geq \frac{1}{2C}.$$

Then

$$\|(T + T^2 + \cdots + T^n)f\|^2 \geq \sum_{j=1}^n \|T^j f\|^2 \geq \frac{n}{4C^2},$$

since all inner products in the expansion of the left side are non-negative. For  $n$  large enough, this inequality contradicts  $\|T + T^2 + \cdots + T^n\| \leq C$ .

□

**Remarks.** 1. The condition  $\sup_{x \in \Omega} T1(x) < +\infty$  can be expressed in a “geometric form”

$$\sup_{x \in \Omega} \int_0^{+\infty} \frac{\omega(B(x, t))}{t^2} dt < +\infty,$$

where  $B(x, t) = \{y \in \Omega : d(x, y) < t\}$ . Indeed,

$$\begin{aligned} T1(x) &= \int_{\Omega} K(x, y) d\omega(y) = \int_{\Omega} \frac{d\omega(y)}{d(x, y)} \\ &= \int_{\Omega} \int_{d(x, y)}^{\infty} \frac{dt d\omega(y)}{t^2} = \int_0^{+\infty} \frac{\omega(B(x, t))}{t^2} dt. \end{aligned}$$

2. The condition  $\sup_{x \in \Omega} T1(x) < +\infty$  can be replaced with  $\|T1\|_{L^\infty(\omega)} < +\infty$ , which in its turn is equivalent to  $\|T\|_{L^1(\omega) \rightarrow L^1(\omega)} < +\infty$ .

Indeed, let  $E = \{x : T1(x) \leq \|T1\|_{L^\infty(\omega)}\}$ . Then  $\omega(\Omega \setminus E) = 0$ , so  $E$  is non-empty. Fix any point  $x \in \Omega$ . Then the following two cases are possible: (i)  $\inf_{y \in E} d(x, y) = 0$ . In this case, for every  $\epsilon > 0$ , there exists  $y \in E$  such that  $d(x, y) < \epsilon$ , and therefore  $d(y, z) \leq \kappa(d(x, z) + \epsilon)$ , whence

$$\int_{\Omega} \frac{d\omega(z)}{d(x, z) + \epsilon} \leq \kappa T1(y) \leq \kappa \|T1\|_{L^\infty(\omega)}.$$

Passing to the limit as  $\epsilon \rightarrow 0$ , we obtain  $T1(x) \leq \kappa \|T1\|_{L^\infty(\omega)}$ .

(ii)  $D = \inf_{y \in E} d(x, y) > 0$ . Then choose any point  $y \in E$  with  $d(x, y) \leq 2D$ . Note that for all  $z \in E$ ,

$$d(y, z) \leq \kappa(d(x, y) + d(x, z)) \leq \kappa(2D + d(x, z)) \leq 3\kappa d(x, z).$$

Thus in this case

$$T1(x) = \int_E \frac{d\omega(z)}{d(x, z)} \leq 3\kappa \int_E \frac{d\omega(z)}{d(y, z)} = 3\kappa T1(y) \leq 3\kappa \|T1\|_{L^\infty(\omega)}.$$

3. Let  $\mathcal{B}$  denote the space of all bounded functions on  $\Omega$  with norm  $\|u\|_{\mathcal{B}} = \sup_{x \in \Omega} |u(x)|$ . Suppose  $T$  is an integral operator with quasi-metric kernel. Clearly  $T : \mathcal{B} \rightarrow \mathcal{B}$  is bounded if and only if  $\sup_{x \in \Omega} T1(x) < +\infty$ . Under this additional assumption, W. Hansen [H] showed that condition (a) of Theorem 3.2 is equivalent to  $r(T)_{\mathcal{B}} < 1$ , where  $r(T)_{\mathcal{B}}$  is the spectral radius of  $T$  in  $\mathcal{B}$ . This result is a consequence of Theorem 3.2 above. Moreover, for operators  $T$  with quasi-metric kernels which are bounded on  $\mathcal{B}$ , we have  $r(T)_{\mathcal{B}} = \|T\|$ .

Indeed,  $\|T\|_{\mathcal{B} \rightarrow \mathcal{B}} \geq \|T\|_{L^\infty(\omega) \rightarrow L^\infty(\omega)} = \|T\|_{L^1(\omega) \rightarrow L^1(\omega)}$ . Using interpolation, and the formula  $r(T)_{\mathcal{B}} = \lim_{n \rightarrow \infty} \|T^n\|_{\mathcal{B} \rightarrow \mathcal{B}}^{1/n}$  we see that  $r(T)_{\mathcal{B}} \geq r(T)_{L^2(\omega)} = \|T\|$ . By an argument similar to that used in the

proof of Theorem 3.2 (with  $\mathcal{B}$ , or  $L^\infty(\omega)$ , in place of  $L^2(\omega)$ ) it follows that (a) implies  $r(T)_\mathcal{B} < 1$ . Thus, the condition  $r(T)_\mathcal{B} < 1$  is equivalent to  $\|T\| < 1$  for operators  $T$  with quasi-metric kernels bounded on  $\mathcal{B}$ . To prove that  $r(T)_\mathcal{B} = \|T\|$ , it remains to notice that, for  $\epsilon > 0$ , the operator  $T_\epsilon = (\|T\| + \epsilon)^{-1}T$  satisfies  $\|T_\epsilon\| < 1$ , and hence  $r(T_\epsilon)_\mathcal{B} < 1$ , which yields  $r(T)_\mathcal{B} < \|T\| + \epsilon$ . Conversely,  $S_\epsilon = (r(T)_\mathcal{B} + \epsilon)^{-1}T$  satisfies  $r(S_\epsilon)_\mathcal{B} < 1$  which gives  $\|S_\epsilon\| < 1$ , that is,  $\|T\| < r(T)_\mathcal{B} + \epsilon$ . Letting  $\epsilon \rightarrow 0$  yields  $r(T)_\mathcal{B} = \|T\|$ .

We will now extend our results to a wider class of quasi-metrically modifiable kernels. It turns out that in many interesting applications the quasi-metric property fails but the quasi-metric modifiability holds. Let  $K$  be a quasi-metrically modifiable kernel on a measure space  $(\Omega, \omega)$ , with modifier  $m$ , so that  $\tilde{K}(x, y) = K(x, y)/(m(x)m(y))$  is a quasi-metric kernel.

We also consider the measure  $d\tilde{\omega} = m^2 d\omega$  and the operator  $\tilde{T}$  defined by

$$\tilde{T}f(x) = \int_{\Omega} \tilde{K}(x, y)f(y) d\tilde{\omega}(y).$$

Various properties of a quasi-metrically modifiable kernel  $K$  and the corresponding integral operator  $T$  can be reduced to those of  $\tilde{K}$  and  $\tilde{T}$ . The following properties are straightforward, and we leave their proofs to the reader.

- (a)  $\tilde{K}_j(x, y) = \frac{K_j(x, y)}{m(x)m(y)}$  for all  $j \in \mathbb{N}$ .
- (b) If  $f \in L^2(\omega)$ , then  $\tilde{f} = \frac{f}{m} \in L^2(\tilde{\omega})$ .
- (c)  $\tilde{T}^j \tilde{f} = \frac{T^j f}{m}$ , for all  $j \in \mathbb{N}$ .
- (d)  $\tilde{T}^j 1 = \frac{T^j m}{m}$ , for all  $j \in \mathbb{N}$ .
- (e)  $\|\tilde{T}\|_{L^2(\tilde{\omega})} = \|T\|_{L^2(\omega)}$ .

Applying Theorem 1.1 to  $\tilde{K}$  and  $\tilde{T}$ , and rewriting the conclusions in terms of  $K$  and  $T$ , we deduce that Theorem 1.1 remains valid verbatim for quasi-metrically modifiable kernels, which we state as the following corollary.

**Corollary 3.3.** *Let  $(\Omega, \omega)$  be a  $\sigma$ -finite measure space, and let  $K$  be a quasi-metrically modifiable kernel on  $\Omega$  with constant  $\kappa$ . Let  $K_1 = K$*

and define  $K_j$  by (1.2) for  $j \geq 2$ . Then there exists  $c > 0$ , depending only on  $\kappa$ , such that (2.1) holds.

Define  $T$  by (1.1). If  $\|T\|_{L^2(\omega) \rightarrow L^2(\omega)} < 1$ , then there exists  $C > 0$ , depending only on  $\kappa$  and  $\|T\|$ , such that

$$\sum_{j=1}^{\infty} K_j(x, y) \leq K(x, y) e^{CK_2(x, y)/K(x, y)}.$$

Theorem 3.1 becomes a statement concerning the minimal positive solution  $u_0$ , defined by  $u_0 = m + \sum_{j=1}^{\infty} T^j m$ , of the equation  $u = Tu + m$ , where  $m$  is the quasi-metric modifier. We obtain estimates for  $u_0$  in the next corollary which is deduced by applying Theorem 3.1 to  $u_0/m = 1 + \sum_{j=1}^{\infty} \tilde{T}^j 1$ .

**Corollary 3.4.** *Suppose  $K$  is a quasi-metrically modifiable kernel with modifier  $m$  and constant  $\kappa$  on  $(\Omega, \omega)$ . Define  $T$  by (1.1). Then there exists  $c > 0$  depending only on  $\kappa$  such that*

$$(3.5) \quad u_0 \geq m e^{c(Tm)/m}.$$

*If  $\|T\| < 1$ , then there exists  $C > 0$  depending only on  $\kappa$  and  $\|T\|$  such that*

$$(3.6) \quad u_0 \leq m e^{C(Tm)/m}.$$

*Moreover,  $u_0 \approx m$  if and only if  $\|T\| < 1$ , and  $Tm \leq Cm$ .*

The next corollary is a direct analogue of Theorem 3.2 for quasi-metrically modifiable kernels, proved by reducing to the quasi-metric case via (a) - (e) above.

**Corollary 3.5.** *Let  $(\Omega, \omega)$  be a  $\sigma$ -finite measure space. Suppose  $K : \Omega \times \Omega \rightarrow (0, +\infty]$  is quasi-metrically modifiable with constant  $\kappa$  and modifier  $m$ , and that  $K$  is not identically  $\infty$ . Let  $T$  be the integral operator corresponding to  $K$  and let  $u_0$  be the minimal positive solution of the equation  $u = Tu + m$ . Then the following statements are equivalent:*

(a) *There exists  $C_1 > 0$  such that  $\sum_{j=1}^{\infty} K_j(x, y) \leq C_1 K(x, y)$  for all  $x, y \in \Omega$ .*

(b)  *$\|T\| < 1$  and  $K_2(x, y) \leq C_2 K(x, y)$  for all  $x, y \in \Omega$ , for some  $C_2 > 0$  (or, equivalently,  $\sup_{x \in \Omega} (Tm(x))/m(x) < +\infty$ ).*

(c)  *$\sup_{x \in \Omega} (u_0(x)/m(x)) < +\infty$ .*

In conclusion of this section we discuss an intrinsic characterization of the class of quasi-metrically modifiable kernels. Recall that a positive

symmetric kernel  $K$  is quasi-metrically modifiable with modifier  $m > 0$  if and only if  $d(x, y) = m(x)m(y)D(x, y)$ , where  $D(x, y) = 1/K(x, y)$ , is a quasi-metric. Then  $D$  satisfies the Ptolemy inequality:

$$D(y_1, y_3)D(y_2, y_4) \leq 4\kappa^2 (D(y_1, y_2)D(y_3, y_4) + D(y_1, y_4)D(y_2, y_3))$$

for all  $y_1, y_2, y_3, y_4 \in \Omega$ . Indeed, by Lemma 2.2, we have such an inequality for  $d$ . Using the relation between  $d$  and  $D$  and cancelling the term  $m(y_1)m(y_2)m(y_3)m(y_4)$  yields the Ptolemy inequality for  $D$ .

On the other hand, suppose  $K$  is a positive, symmetric kernel with the property that for some  $w \in \Omega$ ,  $K(x, w) < \infty$  for all  $x$ , such that  $D = 1/K$  satisfies the Ptolemy inequality

$$D(x, y)D(z, w) \leq C (D(x, z)D(y, w) + D(x, w)D(y, z))$$

for all  $x, y, z \in \Omega$ . Then dividing by  $D(x, w)D(y, w)D(z, w)$  yields

$$\frac{D(x, y)}{D(x, w)D(y, w)} \leq C \left( \frac{D(x, z)}{D(x, w)D(z, w)} + \frac{D(y, z)}{D(y, w)D(z, w)} \right).$$

Hence  $m(x) = 1/D(x, w) = K(x, w)$  is a quasi-metric modifier for  $K$ . More generally, if  $\omega(\{x : K(x, w) = \infty\}) = 0$ , then  $\{x : K(x, w) = \infty\}$  can be deleted from  $\Omega$  without significant effect, and the contrary case is somewhat degenerate. Such observations about quasi-metric modifiers were first noticed by Hansen and Netuka [HN] (Proposition 8.1).

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